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General quadratic gauge theory: constraint structure, symmetries and physical functions

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Abstract

How can we relate the constraint structure and constraint dynamics of the general gauge theory in the Hamiltonian formulation to specific features of the theory in the Lagrangian formulation, especially relate the constraint structure to the gauge transformation structure of the Lagrangian action? How can we construct the general expression for the gauge charge if the constraint structure in the Hamiltonian formulation is known? Whether we can identify the physical functions defined as commuting with first-class constraints in the Hamiltonian formulation and the physical functions defined as gauge invariant functions in the Lagrangian formulation? The aim of the present paper is to consider the general quadratic gauge theory and to answer the above questions for such a theory in terms of strict assertions. To fulfil such a programme, we demonstrate the existence of the so-called superspecial phase-space variables in terms of which the quadratic Hamiltonian action takes a simple canonical form. On the basis of such a representation, we analyse a functional arbitrariness in the solutions of the equations of motion of the quadratic gauge theory and derive the general structure of symmetries by analysing a symmetry equation. We then use these results to identify the two definitions of physical functions and thus prove the Dirac conjecture.

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1. Introduction

Most contemporary particle-physics theories are formulated as gauge theories. It is well known that within the Hamiltonian formulation, gauge theories are theories with constraints (in particular, with first-class constraints (FCC)). This is the main reason for a long intensive study of formal theory of constrained systems. The theory of constrained systems began with pioneer works by Bergmann and Dirac [1, 2] and was then developed and presented in some review books [3–7], it still attracts a lot of attention from researchers. The first steps of the theory in formulating dynamics of constrained systems in the phase space, elaborating the procedure of finding all the constraints (the Dirac procedure), and reorganizing the constraints to the FCC and second-class constraints (SCC) were relatively simple. From the very beginning, it became clear that the presence of FCC among the complete set of constraints in the Hamiltonian formulation is a direct indication that the theory is a gauge one, i.e., its Lagrangian action is invariant under gauge transformations that in the general case are continuous transformations parametrized by arbitrary functions of time (of space–time coordinates in the case of field theory). It was demonstrated that the number of independent gauge parameters is equal to the number μ_1 of primary FCC, and the total number of unphysical variables is equal to the number μ of all FCC, in spite of the fact that the equations of motion contain only μ_1 arbitrary functions of time (undetermined Lagrange multipliers = the primary FCC), see [6] and references therein. At the same time, we proved that for a class of theories for which the constraint structure of the whole theory and its quadratic approximation is the same and for which the constraint structure does not change from point to point in the phase-space (we call such theories perturbative ones), physical functions in the Hamiltonian formulation have to commute with FCC. In a sense, this statement can be identified with the so-called Dirac conjecture. All known, until now, models where the Dirac conjecture does not hold are nonperturbative in the above sense. After this preliminary progress in the theory of constrained systems, it became clear that a natural and very important continuation of the study is to try to relate the constraint structure and constraint dynamics of a gauge theory in the Hamiltonian formulation to specific features of the theory in the Lagrangian formulation, especially to relate the constraint structure to the gauge transformation structure of the Lagrangian action. One of the key problems here is the following: how to construct a general expression for the gauge charge if the constraint structure in the Hamiltonian formulation is known? Another principal question, closely related to the latter one, is: whether we can identify the physical functions defined as commuting with FCC in the Hamiltonian formulation and the physical functions defined as gauge invariant functions in the Lagrangian formulation? Many efforts have been made to attempt to answer these questions (see, e.g., [8]). All previous considerations contain some restrictive assumptions about the theory structure (in particular, about the constraint structure), such that strictly proved answers to all the above questions are still unknown for a general gauge theory (even belonging to the above-mentioned perturbative class).

The aim of the present work is to consider a general quadratic gauge theory and to answer the above questions for such a theory in terms of strict assertions. The motivation is that for the majority of perturbative gauge theories, their behaviour is in essence determined by the quadratic part of the action, and the nonquadratic part is 'small', in a sense. The constraint and gauge structure of the complete theory and its quadratic approximation is the same. Constraints of the complete theory differ from linear constraints of the quadratic theory by 'small' nonlinear terms, such that the number of first-class and second-class constraints remains unchanged. The gauge transformations of the complete theory and of its quadratic approximation have the same number of gauge parameters. The majority of the properties of the complete gauge theory and of its quadratic approximation are the same. However, as was already mentioned above, the consideration of a general gauge theory is sometimes a formidable task. At the same time, the simplifications due to the quadratic approximation allow us to present strict derivations and illustrations of relations between the Hamiltonian and Lagrangian structures of gauge theories. In particular, we establish the relation between the constraint structure of the theory and the structure of its gauge transformations, represent the gauge charge as a decomposition in constraints, prove the Dirac conjecture and identify the physical functions in Hamiltonian and Lagrangian formulations. To fulfil such a programme, we demonstrate the existence of the so-called superspecial phase-space variables (section 2 and the appendix), in terms of which the quadratic Hamiltonian action takes a simple canonical form. On the basis of such a representation, we analyse a functional arbitrariness in the solutions to the equations of motion of the quadratic gauge theory (section 3), and derive a general structure of symmetries by analysing a symmetry equation (section 4). In section 5, we use these results to identify the two definitions of physical functions and thus prove the Dirac conjecture.

2. Superspecial phase-space variables

First, we recall (see [6]) that there exists a canonical transformation from the initial phase-space variables $\eta = (q, p)$ to the special phase-space variables $\vartheta = (\omega, Q, \Omega)$ with the following properties: the constraint surface is described by the equations $\Omega = 0$. The variables Ω are divided into two groups: $\Omega = (\mathcal{P}, U)$, where U are all the SCC and $\mathcal{P}$ are all the FCC. At the same time, $\mathcal{P}$ are the momenta conjugate to the coordinates Q. Moreover, the special variables can be chosen such that $\Omega = (\Omega^{(1)}, \Omega^{(2\ldots)})$, where $\Omega^{(1)}$ are primary and $\Omega^{(2\ldots)}$ are secondary constraints. Respectively, $\Omega^{(1)} = (\mathcal{P}^{(1)}, U^{(1)})$, $\Omega^{(2\ldots)} = (\mathcal{P}^{(2\ldots)}, U^{(2\ldots)})$; $\mathcal{P} = (\mathcal{P}^{(1)}, \mathcal{P}^{(2\ldots)})$, $U = (U^{(1)}, U^{(2\ldots)})$; $\mathcal{P}^{(1)}$ are primary FCC, $\mathcal{P}^{(2\ldots)}$ are secondary FCC, $U^{(1)}$ are primary SCC, $U^{(2)}$ are secondary SCC. The Hamiltonian action S_{H} of a general quadratic gauge theory has the structure

$$
\begin{aligned}
&S_{\mathrm{H}}[\boldsymbol{\vartheta}] = S_{\mathrm{ph}}[\omega] + S_{\text{non-ph}}[\boldsymbol{\vartheta}], \qquad \boldsymbol{\vartheta} = (\vartheta, \lambda), \\
&S_{\mathrm{ph}}[\omega] = \int [\omega_p \dot{\omega}_q - H_{\mathrm{ph}}(\omega)]\mathrm{d}t, \\
&S_{\text{non-ph}}[\boldsymbol{\vartheta}] = \int \left[\mathcal{P}\dot{Q} + U_p \dot{U}_q - H^{(1)}_{\text{non-ph}}(\boldsymbol{\vartheta})\right]\mathrm{d}t,
\end{aligned}
\tag{1}
$$

where

$$
\begin{aligned}
H^{(1)}_{\text{non-ph}} = {}& (Q^{(1)}A + Q^{(2\ldots)}B + \omega C)\mathcal{P}^{(2\ldots)} + \mathcal{P}^{(2\ldots)}D\mathcal{P}^{(2\ldots)} \\
& + \mathcal{P}^{(2\ldots)}EU^{(2\ldots)} + U^{(2\ldots)}GU^{(2\ldots)} + \lambda_{\mathcal{P}}\mathcal{P}^{(1)} + \lambda_U U^{(1)},
\end{aligned}
\tag{2}
$$

and A, B, C, E and G are some matrices (in the general case depending on time). We note that the special variables (ω, Q, Ω) may be chosen in more than one way. The equations of motion are

$$
\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\vartheta}} = 0 \Longrightarrow \begin{cases} \dot{\vartheta} = \{\vartheta, H^{(1)}\} \\ \Omega = 0, \end{cases}
$$

where

$$
H^{(1)} = H_{\mathrm{ph}} + H_{\text{non-ph}}
$$

is the total Hamiltonian. In what follows, we call $\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\vartheta}}$ and $O\left(\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\vartheta}}\right)$ the extremals.

One can demonstrate (see the appendix) that the special phase-space variables can be chosen such that the non-physical part of the total Hamiltonian (2) takes a simple (canonical) form,

$$
H^{(1)}_{\text{non-ph}} = H^{(1)}_{\mathrm{FCC}} + H^{(1)}_{\mathrm{SCC}},
\tag{3}
$$

where

$$
\begin{aligned}
&H^{(1)}_{\mathrm{FCC}} = \sum_{a=1}^{\aleph_\chi} \left(\sum_{i=1}^{a-1} Q^{(i|a)}\mathcal{P}^{(i+1|a)} + \lambda^a_{\mathcal{P}}\mathcal{P}^{(1|a)} \right), \\
&H^{(1)}_{\mathrm{SCC}} = U^{(2\ldots)}FU^{(2\ldots)} + \lambda_U U^{(1)}.
\end{aligned}
$$

Here $(Q, \mathcal{P}) = (Q^{(i|a)}, \mathcal{P}^{(i|a)})$, $\lambda_{\mathcal{P}} = \left(\lambda_{\mathcal{P}}^{a}\right)$, $a = 1, \ldots, \aleph_\chi$, $i = 1, \ldots, aF$ is a matrix, and $\aleph_\chi$ is the number of the stages of the Dirac procedure that is necessary to determine all the independent FCC. In what follows, we call such special phase-space variables the superspecial phase-space variables. In terms of the superspecial phase-space variables, the consistency conditions for the primary FCC $\mathcal{P}^{(1|a)}$, $a > 1$, determine the secondary FCC $\mathcal{P}^{(2|\aleph_\chi)}$, and so on, creating the following a-chain of FCC, $\mathcal{P}^{(1|a)} \to \mathcal{P}^{(2|a)} \to \mathcal{P}^{(3|a)} \ldots \mathcal{P}^{(a|a)}$, see the following scheme:

$$
\begin{array}{ccccccccc}
\mathcal{P}^{(1|\aleph_\chi)} & \to & \mathcal{P}^{(2|\aleph_\chi)} & \to & \cdots & \to & \mathcal{P}^{(\aleph_\chi-1|\aleph_\chi)} & \to & \mathcal{P}^{(\aleph_\chi|\aleph_\chi)} \\
\mathcal{P}^{(1|\aleph_\chi-1)} & \to & \mathcal{P}^{(2|\aleph_\chi-1)} & \to & \cdots & \to & \mathcal{P}^{(\aleph_\chi-1|\aleph_\chi-1)} & & \\
\vdots & \vdots & \vdots & \vdots & \vdots & & & & \\
\mathcal{P}^{(1|2)} & \to & \mathcal{P}^{(2|2)} & & & & & & \\
\mathcal{P}^{(1|1)}. & & & & & & & &
\end{array}
$$

The consistency conditions for the constraints $\mathcal{P}^{(a|a)}$, $a = 1, \ldots, \aleph_\chi$ do not yield any new constraints. We note that in the canonical form the non-physical part of the total Hamiltonian is independent of the coordinates $Q^{(a|a)}$.

3. Functional arbitrariness in solutions of equations of motion

In theories with FCC, the equations of motion do not determine a unique trajectory for given initial data. In what follows, we study this problem for the quadratic gauge theories using the superspecial phase-space variables. The equations of motion that follow from actions (1) and (2), with taking (3) into account, are

$$
\dot{\omega} = \{\omega, H_{\mathrm{ph}}\}, \qquad \Omega = 0, \tag{4}
$$

and

$$
\dot{Q}^{(i|a)} = \{Q^{(i|a)}, H_{\text{non-ph}}\} \Longrightarrow \begin{cases} \dot{Q}^{(1|a)} = \lambda_{\mathcal{P}}^{a}, \\ \dot{Q}^{(2|a)} = Q^{(1|a)}, \\ \cdots \\ \dot{Q}^{(a|a)} = Q^{(a-1|a)}. \end{cases} \tag{5}
$$

We see that equations (4) for the physical variables ω and for Ω have a unique solution whenever initial data for these variables are given. There exists a functional arbitrariness in solutions to the equations of motion (5) for the variables Q, because these equations contain arbitrary functions of time $\lambda_{\mathcal{P}}(t)$. We note that the number of variables Q is equal to the number of all FCC and, in general, this number is larger than the number of the arbitrary functions $\lambda_{\mathcal{P}}(t)$. However, as will be seen below, because of the specific structure of the equations, the 'influence' of these arbitrary functions on solutions for Q is very strong. This fact is extremely important for the physical interpretation of the variables Q and for the general physical interpretation of theories with FCC. The following proposition describes to what extent the variables Q are affected by the arbitrary functions $\lambda_{\mathcal{P}}(t)$.

The equations of motion (5) for the variables Q are completely controllable[3] by the functions $\lambda_{\mathcal{P}}(t)$. In the case under consideration, this means that under a proper choice of the functions $\lambda_{\mathcal{P}}^a(t)$, equations (5) have a solution with the properties

$$\begin{aligned} &Q^{(i|a)}|_{t=0} = 0, \qquad && Q^{(i|a)}|_{t=\tau} = \Delta^{(i|a)}, \qquad && i = 1, \ldots, a, \\ &\left.\frac{\mathrm{d}^s \lambda_{\mathcal{P}}^a}{\mathrm{d}^s t}\right|_{t=0} = 0, \qquad && \left.\frac{\mathrm{d}^s \lambda_{\mathcal{P}}^a}{\mathrm{d}^s t}\right|_{t=\delta} = \delta_{(s)}^a, \qquad && s = 0, 1, \ldots, K, \end{aligned} \tag{6}$$

where τ, $\Delta^{(i|a)}$, $\Delta^{(i|a)}$, $\delta_{(s)}^a$ and the integer K are arbitrary.

Because of the simple structure of the equations of motion in superspecial phase-space variables, the proof of the above assertion can be done in a constructive manner. Namely, we explicitly present such a solution. It has the form

$$Q^{(i|a)} = \frac{\mathrm{d}^{a-i} X^a}{\mathrm{d}t^{a-i}}, \qquad i = 1, \ldots, a,$$

if we choose

$$\lambda_{\mathcal{P}}^a = \frac{\mathrm{d}^a X^a}{\mathrm{d}t^a},$$

where $X^a(t)$ are arbitrary smooth functions obeying the following boundary conditions:

$$\left.\frac{\mathrm{d}^s X^a}{\mathrm{d}t^s}\right|_{t=0} = 0, \qquad s = 0, \ldots, K + a,$$

$$\left.\frac{\mathrm{d}^s X^a}{\mathrm{d}t^s}\right|_{t=\tau} = \begin{cases} Q^{(a-s|a)}|_{t=\tau} = \Delta^{(a-s|a)}, & s = 0, \ldots, a-1, \\ \left.\dfrac{\mathrm{d}^{s-a} \lambda_{\mathcal{P}}^a}{\mathrm{d}t^{s-a}}\right|_{t=\tau} = \delta_{(s-a)}^a, & s = a, \ldots, K + a. \end{cases}$$

For example, the functions $X^a(t)$ can be chosen as

$$X^a(t) = f(t) \left[\sum_{s=0}^{a-1} \frac{1}{s!} \Delta^{(a-s|a)} (t - \delta)^s + \sum_{s=a}^{K+a} \frac{1}{s!} \delta_{(s-a+1)}^a (t - \delta)^s \right]$$

where $f(t)$ is an arbitrary smooth function that is respectively equal to zero and to one in the neighbourhoods of the points $t = 0$ and $t = \tau$. An example of such a function is[4]

$$\begin{aligned} &f(t) = \begin{cases} 0, & t \leqslant \varepsilon, \\ \dfrac{1}{1 + \mathrm{e}^u}, & u = \dfrac{1}{t - \varepsilon} + \dfrac{1}{t - (\tau - \varepsilon)}, \qquad \varepsilon \leqslant t \leqslant \tau - \varepsilon, \\ 1, & t \geqslant \tau - \varepsilon, \end{cases} \\ &\lim_{t \to \varepsilon + 0} f^{[s]}(t) = 0, \qquad \lim_{t \to \tau - \varepsilon - 0} f^{[s]}(t) = \delta_{0,s}, \qquad s \geqslant 0. \end{aligned} \tag{7}$$

The proved proposition is crucial for the understanding of the structure of theories with FCC (gauge theories) and for their physical interpretation. The most remarkable fact is the following: the functional arbitrariness in equations of motion of theories with FCC (gauge theories) is due to the undetermined Lagrange multipliers to the primary FCC. However, this arbitrariness affects essentially more variables. In the special variables, all the variables Q are controllable by the undetermined Lagrange multipliers. The number of Q-variables is equal to the number of all the FCC and is greater than the number of the Lagrange multipliers.

[3] For an exact definition of the controllability see, e.g., the book [9].

[4] Here and in what follows, we use the notation

$$f^{[s]} = \frac{\mathrm{d}^s f}{\mathrm{d}t^s}.$$

4. Symmetries

We recall that a transformation $q(t) \to q'(t)$ is called a symmetry of an action S if

$$L(q, \dot{q}) \to L'(q, \dot{q}) = L(q, \dot{q}) + \frac{\mathrm{d}F}{\mathrm{d}t}, \tag{8}$$

where F is a local function. In what follows, only infinitesimal symmetry transformations $q \to q + \delta q$ with local functions δq are considered. These symmetry transformations can be global, gauge and trivial ones. Gauge transformations are parametrized by some arbitrary functions of time, gauge parameters (in the case of a field theory the gauge parameters depend on all space–time variables). Any infinitesimal symmetry transformation implies a conservation low (Nöether theorem):

$$\begin{aligned} &\frac{\mathrm{d}G}{\mathrm{d}t} = -\delta q^a \frac{\delta S}{\delta q^a} \Longrightarrow G = \text{const on extremals}, \\ &G = P - F, \qquad P = \frac{\partial L}{\partial \dot{q}^a}\delta q^a, \qquad \delta L = \frac{\mathrm{d}F}{\mathrm{d}t}. \end{aligned} \tag{9}$$

The local function G is referred to as the conserved charge related to the symmetry δq of the action S. The quantities δq, S and G are related by equation (9). In what follows, we call this equation the symmetry equation. The symmetry equation for the Hamiltonian action $S_{\mathrm{H}}[\boldsymbol{\vartheta}]$ has the form

$$\delta\boldsymbol{\vartheta}\frac{\delta S_{\mathrm{H}}}{\delta\boldsymbol{\vartheta}} + \frac{\mathrm{d}G}{\mathrm{d}t} = 0. \tag{10}$$

4.1. Trivial symmetries

For any action, there are trivial symmetry transformations,

$$\delta_{\mathrm{tr}} q^a = \hat{U}^{ab}\frac{\delta S}{\delta q^b}, \tag{11}$$

where $\hat{U}$ is an antisymmetric local operator, that is $(\hat{U}^T)^{ab} = -\hat{U}^{ab}$. The trivial symmetry transformations do not affect genuine trajectories. Using the simple action structure in superspecial phase-space variables, we can prove the following assertion: for theories with FCC, symmetries of the Hamiltonian action that vanish on the extremals are trivial symmetries.

To prove this assertion we consider the Hamiltonian action $S_{\mathrm{H}}[\boldsymbol{\vartheta}]$, $\boldsymbol{\vartheta} = (\vartheta, \lambda)$ (see equation (1), and equation (A.12) from the appendix) of a theory with FCC in the superspecial phase-space variables ϑ. We can see that the equations of motion

$$\frac{\delta S_{\mathrm{H}}}{\delta \mathcal{U}} = 0, \qquad \frac{\delta S_{\mathrm{H}}}{\delta \mathcal{Q}} = 0, \qquad \frac{\delta S_{\mathrm{H}}}{\delta \mathcal{P}} = 0,$$

where $\mathcal{Q} = (\lambda^a_{\mathcal{P}}, Q^{(i|a)}, i = 1, \ldots, a-1, a = 1, \ldots, \aleph_\chi)$, have solutions of the form $\mathcal{U} = \mathcal{P} = 0$ and $\mathcal{Q} = \psi(Q^{(i|i)})$, where ψ are local functions of the indicated arguments. Therefore, the variables $\mathcal{U}$, $\mathcal{P}$ and $\mathcal{Q}$ are auxiliary ones[5]. Excluding these variables from the action S_{H}, we obtain a dynamically equivalent action $\bar{S}_{\mathrm{H}}[\omega, Q^{(i|i)}]$. Taking into account that $\mathcal{U} = \mathcal{P} = 0 \Longrightarrow \Omega = 0$, and the relation $S_{\mathrm{H}}|_{\Omega=0} = S_{\mathrm{ph}}[\omega]$, we find

$$\bar{S}_{\mathrm{H}}[\omega, Q^{(i|i)}] = S_{\mathrm{H}}|_{\mathcal{U}=\mathcal{P}=0, \mathcal{Q}=\psi} = S_{\mathrm{ph}}[\omega].$$

[5] Suppose an action $S[q, y]$ contains two groups of coordinates q and y such that the coordinates y can be expressed as local functions $y = \bar{y}(q^{[l]}, l < \infty)$ of q and their time derivatives by the help of the equations $\delta S/\delta y = 0$. We call y the auxiliary coordinates. The action $S[q, y]$ and the reduced action $S[q] = S[q, \bar{y}]$ lead to the same equations for the coordinates q, see [11, 12]. The actions $S[q, y]$ and $S[q]$ are called dynamically equivalent actions. One can prove that there exists a one-to-one correspondence (isomorphism) between the symmetry classes of the extended action. Symmetries are equivalent if they differ by a trivial transformation.

Let a transformation $\delta\boldsymbol{\vartheta}$ which vanishes on the extremals be a symmetry of the action S_{H}. Consider the reduced transformation $\bar{\delta}\omega, \bar{\delta}Q^{(i|i)}$,

$$(\bar{\delta}\omega, \bar{\delta}Q^{(i|i)}) = \delta\boldsymbol{\vartheta}|_{\mathcal{U}=\mathcal{P}=0, \mathcal{Q}=\psi}.$$

It is evident that the reduced transformation vanishes on the extremals of the reduced action $\bar{S}_{\mathrm{H}}$ and is a symmetry transformation of the action $\bar{S}_{\mathrm{H}}$. This implies that

$$\bar{\delta}\omega = \hat{m}\frac{\delta S_{\mathrm{ph}}}{\delta\omega}, \qquad \bar{\delta}Q^{(i|i)} = \left(\hat{n}\frac{\delta S_{\mathrm{ph}}}{\delta\omega}\right)^{(i|i)},$$

where $\hat{m}$ and $\hat{n}$ are some local operators. The transformation $\bar{\delta}\omega$ is obviously the symmetry transformation of the nonsingular action S_{ph}, and in addition, this symmetry transformation vanishes on the extremals. We can prove that such a symmetry is always a trivial one and, therefore, $\hat{m}$ is antisymmetric. Therefore, the complete transformation $\bar{\delta}\omega, \bar{\delta}Q^{(i|i)}$ can be represented as

$$\begin{pmatrix} \bar{\delta}\omega \\ \bar{\delta}Q^{(|)} \end{pmatrix} = \hat{M}\begin{pmatrix} \frac{\delta\bar{S}_{\mathrm{H}}}{\delta\omega} \\ \frac{\delta\bar{S}_{\mathrm{H}}}{\delta Q^{(|)}} \end{pmatrix}, \qquad \hat{M} = \begin{pmatrix} \hat{m} & -\hat{n}^T \\ \hat{n} & 0 \end{pmatrix}.$$

The matrix $\hat{M}$ is evidently antisymmetric. Finally, the transformation $\bar{\delta}\omega, \bar{\delta}Q^{(i|i)}$ is a trivial symmetry of the action $\bar{S}_{\mathrm{H}}$. This implies that the extended transformation $\delta\boldsymbol{\vartheta}$ is a trivial symmetry of the action S_{H}.

4.2. Gauge symmetries

We are now going to prove the following assertion: in theories with FCC, there exist nontrivial symmetries $\delta\boldsymbol{\vartheta}$ of the Hamiltonian action S_{H} that are gauge transformations. These symmetries are parametrized by the gauge parameters ν. These parameters are arbitrary functions of time t. Moreover, they can be arbitrary local functions of $\boldsymbol{\vartheta} = (\vartheta, \lambda)$.

The corresponding conserved charge (the gauge charge) is a local function[6] $G = G(\mathcal{P}, \nu^{[]})$, which vanishes on the extremals. The gauge charge has the following decomposition with respect to the FCC:

$$G = \sum_{i=1}^{\aleph_\chi} \nu_{(a)}\mathcal{P}^{(a|a)} + \sum_{i=1}^{\aleph_\chi - 1}\sum_{a=i+1}^{\aleph_\chi} C_{i|a}\mathcal{P}^{(i|a)}. \tag{12}$$

Here $C_{i|a}(\nu^{[]})$ are some local functions, which can be determined from the symmetry equation in an algebraic way, and $\nu = (\nu_{(a)}), \nu_{(a)} = \left(\nu_{(a)}^{\mu_a}\right), a = 1, \ldots, \aleph_\chi$. Here $\nu_{(a)}$ are gauge parameters related to the FCC in a chain whose number is a. The number of gauge parameters $\nu_{(a)}$ is equal to the number of primary FCC in the chain a. The index μ_a labels constraints (and gauge parameters) inside the chain.

The total number of gauge parameters is equal to the number of primary FCC. The total number of independent gauge parameters together with their time derivatives that enter essentially in the gauge charge is equal to the number of all the FCC.

The gauge charge is the generating function for the variations $\delta\vartheta$ of the phase-space variables,

$$\delta\vartheta = \{\vartheta, G\}. \tag{13}$$

The variations $\delta\lambda_{\mathcal{U}}$ are vanishing, and $\delta\lambda_{\mathcal{P}}^a = \nu_{(a)}^{[a]}$.

[6] A local function depends on some variables $x^{a[l]}, a = 1, \ldots, n, l = 0, 1, \ldots, N$ up to some finite order N. We use the following notation for the local functions:

$$F(x^a, x^{a[1]}, x^{a[2]}, \ldots) = F(x^{[]}).$$

To prove the above assertion, we consider the symmetry equation (9) for the case under consideration. Taking the action structure (1), (3), the anticipated form of the gauge charge (12), and of the variations $\delta\vartheta$, into account, we can rewrite this equation as follows:

$$\hat{H}G - \sum_{a=1}^{\aleph_\chi} \lambda_{\mathcal{P}}^a \{\mathcal{P}^{(1|a|)}, G\} = U^{(1)}\delta\lambda_U + \sum_{a=1}^{\aleph_\chi} \delta\lambda_{\mathcal{P}}^a \mathcal{P}^{(1|a)}, \tag{14}$$

where

$$\hat{H}G = \{G, H\} + \left(\frac{\partial}{\partial t} + \nu^{[m+1]}\frac{\partial}{\partial \nu^{[m]}}\right) G,$$
$$H = H_{\mathrm{ph}}(\omega) + \sum_{a=1}^{\aleph_\chi}\sum_{i=1}^{a-1} Q^{(i|a)}\mathcal{P}^{(i+1|a)} + U^{(2\ldots)} F U^{(2\ldots)}. \tag{15}$$

The following commutation relations:

$$\{\mathcal{P}^{(i|a)}, H\} = -\mathcal{P}^{(i+1|a)}, \qquad i = 1, \ldots, \aleph_\chi - 1, \quad a = i+1, \ldots, \aleph_\chi,$$
$$\{\mathcal{P}^{(i|i)}, H\} = 0, \qquad i = 1, \ldots, \aleph_\chi, \quad \{\mathcal{P}, \Omega\} = 0.$$

hold. Equation (14) implies the following equations for the functions $C_{i|a}$ and for the variations $\delta\lambda$:

$$\sum_{i=1}^{\aleph_\chi - 1}\sum_{a=i+1}^{\aleph_\chi}\left[-\mathcal{P}^{(i+1|a)} + \mathcal{P}^{(i|a)}\left(\frac{\partial}{\partial t} + \nu^{[m+1]}\frac{\partial}{\partial\nu^{[m]}}\right)\right] C_{i|a}$$
$$+ \sum_{a=1}^{\aleph_\chi} \mathcal{P}^{(a|a)}\dot{\nu}_{(a)} = U^{(1)}\delta\lambda_U + \sum_{a=1}^{\aleph_\chi}\mathcal{P}^{(1|a)}\delta\lambda_{\mathcal{P}}^a. \tag{16}$$

Considering equation (16) on the constraint surface $\mathcal{P}^{(i|a)} = 0, i = 1, \ldots, \aleph_\chi - 1, a = i, \ldots, \aleph_\chi, U^{(1)} = 0$, we can choose $C_{\aleph_\chi - 1|\aleph_\chi} = \dot{\nu}_{(\aleph_\chi)}$. Substituting this $C_{\aleph_\chi-1|\aleph_\chi}$ into equation (16), we obtain that

$$\sum_{i=1}^{\aleph_\chi - 2}\sum_{a=i+1}^{\aleph_\chi}\left[-\mathcal{P}^{(i+1|a)} + \mathcal{P}^{(i|a)}\left(\frac{\partial}{\partial t} + \nu^{[m+1]}\frac{\partial}{\partial\nu^{[m]}}\right)\right] C_{i|a}$$
$$+ \mathcal{P}^{(\aleph_\chi - 1|\aleph_\chi)}\nu_{\aleph_\chi}^{[2]} + \sum_{a=1}^{\aleph_\chi - 1} \mathcal{P}^{(a|a)}\dot{\nu}_{(a)} = U^{(1)}\delta\lambda_U + \sum_{a=1}^{\aleph_\chi}\mathcal{P}^{(1|a)}\delta\lambda_{\mathcal{P}}^a. \tag{17}$$

Considering equation (17) on the constraint surface $\mathcal{P}^{(i|a)} = 0, i = 1, \ldots, \aleph_\chi - 2, a = i, \ldots, \aleph_\chi, U^{(1)} = 0$, we choose $C_{\aleph_\chi-2|\aleph_\chi-1} = \dot{\nu}_{(\aleph_\chi - 1)}, C_{\aleph_\chi - 2|\aleph_\chi} = \ddot{\nu}_{(\aleph_\chi)}$. We see that we can similarly determine all the $C_{i|a}$ such that

$$C_{i|a} = \nu_{(a)}^{[a-i]}, \qquad i = 1, \ldots, \aleph_\chi - 1, \quad a = i+1, \ldots, \aleph_\chi. \tag{18}$$

Therefore, in the case under consideration, the gauge charge has the following form:

$$G = \sum_{i=1}^{\aleph_\chi}\sum_{a=i}^{\aleph_\chi} \nu_{(a)}^{[a-i]}\mathcal{P}^{(i|a)}. \tag{19}$$

The form of the variations $\delta\vartheta$ follows from (13),

$$\delta Q^{(i|a)} = \nu_{(a)}^{[a-i]}, \qquad \delta\omega = \delta\Omega = 0. \tag{20}$$

After all the $C_{i|a}$ are known, the variations $\delta\lambda$ can be determined from equation (16),

$$\delta\lambda_U = 0, \qquad \delta\lambda_{\mathcal{P}}^a = \nu_{(a)}^{[a]}, \tag{21}$$

which proves the assertion.

5. Structure of arbitrary symmetry

Analysing the symmetry equation, we are going to prove that: any symmetry $\delta\boldsymbol{\vartheta}$ and G of the action S_{H} can be presented as the sum of three types of symmetries

$$\begin{pmatrix}\delta\boldsymbol{\vartheta}\\ G\end{pmatrix} = \begin{pmatrix}\delta_c\boldsymbol{\vartheta}\\ G_c\end{pmatrix} + \begin{pmatrix}\delta_g\boldsymbol{\vartheta}\\ G_g\end{pmatrix} + \begin{pmatrix}\delta_{\mathrm{tr}}\boldsymbol{\vartheta}\\ G_{\mathrm{tr}}\end{pmatrix}, \tag{22}$$

such that:

- The set $\delta_c\boldsymbol{\vartheta}$ and G_c is a global symmetry, canonical for the phase-space variables ϑ. All the variations $\delta_c\boldsymbol{\vartheta}$ and the corresponding conserved charge G_c are either identically zero or do not vanish on the extremals.
- The set $\delta_g\boldsymbol{\vartheta}$ and G_g is a particular gauge transformation given by equations (19), (20) and (21) with specific fixed gauge parameters (i.e., specific fixed forms of the functions $\nu = \bar{\nu}(t, \eta^{[]}, \lambda^{[]})$) that are either identically zero or do not vanish on the extremals. In the latter case, the corresponding conserved charge G_g vanishes on the extremals, whereas the variations $\delta_g\boldsymbol{\vartheta}$ do not.
- The set $\delta_{\mathrm{tr}}\boldsymbol{\vartheta}$ and G_{tr} is a trivial symmetry. All the variations $\delta_{\mathrm{tr}}\boldsymbol{\vartheta}$ and the corresponding conserved charge G_{tr} vanish on the extremals. The charge G_{tr} depends on the extremals quadratically.

In what follows, we present a constructive way for finding the components of decomposition (22).

5.1. Constructing the global canonical part of a symmetry

In what follows, we use the local functions I,

$$I = (\Omega, J) = O\left(\frac{\delta S_{\mathrm{H}}}{\delta\boldsymbol{\vartheta}}\right), \qquad J = (\mathcal{I}, \lambda_U), \qquad \mathcal{I} = \dot{\vartheta} - \{\vartheta, H\} - \{\vartheta, \mathcal{P}^{(1)}\}\lambda_{\mathcal{P}},$$

for the complete set of extremals. One can easily verify that the set I is equivalent to the complete set of the extremals $\delta S_{\mathrm{H}}/\delta\boldsymbol{\vartheta}$. Assuming that $\delta\boldsymbol{\vartheta}$ and G is a symmetry, and taking the structure of the total Hamiltonian in the case under consideration into account, we can write the symmetry equation (10) as

$$\delta\vartheta E^{-1}\mathcal{I} - U^{(1)}\delta\lambda_U - \lambda_U\delta U^{(1)} - \mathcal{P}^{(1)}\delta\lambda_{\mathcal{P}} + \frac{\mathrm{d}G}{\mathrm{d}t} = 0. \tag{23}$$

We denote via $\delta_J\boldsymbol{\vartheta}, G'_J$ the corresponding zero-order terms in the decomposition of the quantities $\delta\boldsymbol{\vartheta}, G$ with respect to the extremals J. We have

$$\begin{pmatrix}\delta\boldsymbol{\vartheta}\\ G\end{pmatrix} = \begin{pmatrix}\delta_J\boldsymbol{\vartheta}\big(\eta, \lambda_{\mathcal{P}}^{[]}\big) + O(J)\\ G'_J\big(\eta, \lambda_{\mathcal{P}}^{[]}\big) + B_m\big(\omega, \lambda_{\mathcal{P}}^{[]}\big)J^{[m]} + O(J^2)\end{pmatrix}. \tag{24}$$

We then rewrite equation (23) retaining only the terms of zero and first order with respect to the extremals J. We obtain

$$\begin{aligned}\delta_J\vartheta E^{-1}\mathcal{I} - \mathcal{P}^{(1)}\delta\lambda_{\mathcal{P}} &= -\hat{H}G'_J + \{\mathcal{P}^{(1)}, G'_J\}\lambda_{\mathcal{P}} + \lambda_U\{U^{(1)}, G'_J\}\\ &\quad + \{\vartheta, G'_J\}E^{-1}\mathcal{I} - J^{[m]}\hat{H}B_m + \lambda_{\mathcal{P}}\{\mathcal{P}^{(1)}, B_m\}J^{[m]} - B_m J^{[m+1]} + O(\Omega I).\end{aligned} \tag{25}$$

Here, the contributions from the terms $U^{(1)}\delta\lambda_U$ and $\delta\lambda_U U^{(1)}$ are accumulated in the term $O(\Omega I)$, and the operator $\hat{H}$ is defined by

$$\hat{H}F = \{F, H\} + \left(\frac{\partial}{\partial t} + \lambda^{[m+1]}\frac{\partial}{\partial\lambda^{[m]}}\right)F. \tag{26}$$

Analysing terms with the extremals $J^{[m]}$ (beginning with the highest derivative) in equation (25), we can see that all $B_m = 0$. Considering then terms proportional to $\mathcal{I}$ in equation (25), we obtain the expression

$$\delta_J \vartheta = \{\vartheta, G'_J\} + O(\Omega)$$

for the variation $\delta_J \vartheta$. Then taking into account that symmetry variations of extremals are proportional to extremals, we derive the following relations:

$$\delta\Omega = O(I) \Longrightarrow \delta_J \Omega = O(\Omega) = \{\Omega, G'_J\} + O(\Omega).$$

Thus, we have

$$\{\Omega, G'_J\} = O(\Omega). \tag{27}$$

We can verify that $\{\mathcal{P}, G'_J\}$ is a first-class function, which means that

$$\{\mathcal{P}, G'_J\} = O(\mathcal{P}) + O(\Omega^2). \tag{28}$$

Considering the remaining terms in equation (25), we obtain the equation

$$\mathcal{P}^{(1)}\delta_J\lambda_{\mathcal{P}} = \hat{H}G'_J + \lambda_{\mathcal{P}}\{\mathcal{P}^{(1)}, G'_J\} + O(\Omega^2), \tag{29}$$

which relates $\delta_J\lambda_{\mathcal{P}}$ and G'_J.

Equation (29) allows the function G'_J to be studied in more detail. For this, we rewrite this equation (taking (26) and (28) into account) as

$$\{G'_J, H\} + \left(\frac{\partial}{\partial t} + \lambda_{\mathcal{P}}^{[m+1]}\frac{\partial}{\partial\lambda_{\mathcal{P}}^{[m]}}\right)G'_J = O(\mathcal{P}) + O(\Omega^2).$$

Analysing the terms which contain Lagrange multipliers $\lambda_\chi^{[m]}$ (beginning with the highest derivative) in this equation, we can see that these multipliers can enter only the terms that vanish on the constraint surface. For example, considering terms with the highest derivative $\lambda_{\mathcal{P}}^{[M+1]}$ in the latter equation, we get

$$\frac{\partial G'_J}{\partial\lambda_{\mathcal{P}}^{[M]}} = O(\mathcal{P}) + O(\Omega^2) \Longrightarrow G'_J = G'_J(\lambda_{\mathcal{P}}, \ldots, \lambda_{\mathcal{P}}^{[M-1]}) + O(\mathcal{P}) + O(\Omega^2).$$

In the same manner we finally obtain: $G'_J = G_J(\vartheta) + O(\mathcal{P}) + O(\Omega^2)$. With equations (27) and (28) taken into account, this implies

$$\{U, G_J\} = O(\Omega), \qquad \{\mathcal{P}, G_J\} = O(\mathcal{P}) + O(\Omega^2).$$

Therefore, the above consideration allows a refined version of representation (24)

$$\begin{pmatrix} \delta\vartheta \\ \delta\lambda_U \\ \delta\lambda_{\mathcal{P}} \\ G \end{pmatrix} = \begin{pmatrix} \{\vartheta, G_J + B_{\mathcal{P}}\mathcal{P}\} + O(I) \\ O(I) \\ \delta_J\lambda_{\mathcal{P}}\big(\vartheta, \lambda_{\mathcal{P}}^{[]}\big) + O(J) \\ G_J + B_{\mathcal{P}}\mathcal{P} + O(I^2) \end{pmatrix}, \tag{30}$$

where $B_{\mathcal{P}} = B_{\mathcal{P}}\big(\vartheta, \lambda_{\mathcal{P}}^{[]}\big)$, and the function $G_J(\vartheta)$ obeys the relations

$$\{G_J, U\} = O(\Omega), \qquad \{G_J, \mathcal{P}\} = O(\mathcal{P}) + O(\Omega^2), \qquad \{G_J, H\} = O(\mathcal{P}) + O(\Omega^2). \tag{31}$$

We select from the function G_J a part G_I that does not vanish on the constraint surface,

$$G_J(\vartheta) = g(\omega) + g_1(\omega, Q)Q + O(\Omega). \tag{32}$$

Because of relation (31), the function $g_1(\omega, Q)$ in (32) is zero, and, moreover, $O(\Omega) = O(\mathcal{P}) + O(\Omega^2)$. We define $G_I(\vartheta)$ as

$$G_I(\vartheta) = g(\omega). \tag{33}$$

We then have

$$G_J(\vartheta) = G_I(\vartheta) + G_1(\vartheta), \qquad G_1(\vartheta) = O(\mathcal{P}) + O(\Omega^2).$$

Therefore, in virtue of (31),

$$G = G_I(\vartheta) + O(\mathcal{P}) + O(I^2), \qquad \hat{H} G_I = 0, \qquad \{U, G_I\} = \{\mathcal{P}, G_I\} = 0. \tag{34}$$

We now define the variations $\delta_I \boldsymbol{\vartheta}$ as

$$\delta_I \vartheta = \{\vartheta, G_I\} \Longrightarrow \delta_I \omega = \{\omega, G_I\}, \qquad \delta_I Q = \delta_I \mathcal{P} = \delta_I U = 0, \qquad \delta_I \lambda_U = \delta_I \lambda_{\mathcal{P}} = 0. \tag{35}$$

The set $\delta_I \boldsymbol{\vartheta}, G_I$ is an exact symmetry of the action S_{H}. In what follows, this symmetry is denoted by

$$\delta_I \vartheta = \delta_c \vartheta = \{\vartheta, G_c\}, \qquad \delta_I \lambda = \delta_c \lambda = 0, \qquad G_I = G_c = g(\omega).$$

5.2. *Constructing the gauge and the trivial parts of a symmetry*

At this step we represent a symmetry $\delta\boldsymbol{\vartheta}, G$ as

$$\delta\boldsymbol{\vartheta} = \delta_c \boldsymbol{\vartheta} + \delta_r \boldsymbol{\vartheta}, \qquad G = G_c + G_r. \tag{36}$$

Because $\delta_c \boldsymbol{\vartheta}, G_c$ is a symmetry, it is obvious that $\delta_r \boldsymbol{\vartheta}, G_r$ is also a symmetry. Using equations (31), we can verify that the following relations,

$$\begin{aligned} G_r &= \sum_{i=1}^{\aleph_{\mathcal{P}}} \sum_{a=i}^{\aleph_{\mathcal{P}}} K_{i|a}\big(\omega, Q, \lambda_{\mathcal{P}}^{[]}\big) \mathcal{P}^{(i|a)} + O(I^2), \\ \delta_r \eta &= \sum_{i=1}^{\aleph_{\mathcal{P}}} \sum_{a=i}^{\aleph_{\mathcal{P}}} \{\eta, \mathcal{P}^{(i|a)}\} K_{i|a}\big(\omega, Q, \lambda_{\mathcal{P}}^{[]}\big) + O(I), \end{aligned} \tag{37}$$

where K are some local functions, hold.

In turn, we represent the symmetry $\delta_r \boldsymbol{\vartheta}, G_r$ in the following form,

$$\delta_r \boldsymbol{\vartheta} = \delta_{\bar{\nu}} \boldsymbol{\vartheta} + \delta_{\mathrm{tr}} \boldsymbol{\vartheta}, \qquad G_r = G_{\bar{\nu}} + G_{\mathrm{tr}}, \tag{38}$$

where the set $\delta_{\bar{\nu}} \boldsymbol{\vartheta}, G_{\bar{\nu}}$ is the gauge transformation given by equations (12) and (13) with specific fixed values of the gauge parameters,

$$\nu_i = \bar{\nu}_i(t, \eta, \lambda^{[]}) = K_{i|i}\big(\omega, Q, \lambda_{\mathcal{P}}^{[]}\big), \tag{39}$$

that are either identically zero or do not vanish on the constraint surface. This implies that

$$G_{\bar{\nu}} = O(\mathcal{P}) + O(I^2), \qquad \delta_{\bar{\nu}} \vartheta = \{\vartheta, G_{\bar{\nu}}\}. \tag{40}$$

We must emphasize that by construction, the functions $K_{i|i}$ (and therefore the gauge transformations) are identically zero whenever they vanish on the constraint surface.

It follows from equations (37) that $\delta_{\mathrm{tr}} \boldsymbol{\vartheta}, G_{\mathrm{tr}}$ is a symmetry whose charge is of the form

$$G_{\mathrm{tr}} = G'_{\mathrm{tr}} + O(I^2), \qquad G'_{\mathrm{tr}} = \sum_{i=1}^{\aleph_{\mathcal{P}}-1} \sum_{a=i+1}^{\aleph_{\mathcal{P}}} K_{i|a}\big(\omega, Q, \lambda_{\mathcal{P}}^{[]}\big) \mathcal{P}^{(i|a)}.$$

In what follows, we will see that $\delta_{\mathrm{tr}} \boldsymbol{\vartheta}, G_{\mathrm{tr}}$ is a trivial symmetry. For the symmetry $\delta_{\mathrm{tr}} \boldsymbol{\vartheta}, G_{\mathrm{tr}}$ we write a decomposition of form (24),

$$\begin{pmatrix} \delta \boldsymbol{\vartheta}_{\mathrm{tr}} \\ G_{\mathrm{tr}} \end{pmatrix} = \begin{pmatrix} \delta_{\mathrm{tr}J} \boldsymbol{\vartheta}\big(\omega, Q, \lambda_{\mathcal{P}}^{[]}\big) + O(J) \\ G'_{\mathrm{tr}J}\big(\vartheta, \lambda_{\mathcal{P}}^{[]}\big) + O(J^2), \; G'_{\mathrm{tr}J} = G'_{\mathrm{tr}} + O(\Omega^2) \end{pmatrix}, \tag{41}$$

taking into account that $B_m = O(\Omega)$. All the relations that hold for the quantities $\delta_J\boldsymbol{\vartheta}, G_J$ also hold for the quantities $\delta_{\mathrm{tr}J}\boldsymbol{\vartheta}, G'_{\mathrm{tr}J}$. In particular, the charge G'_{tr} obeys the equation

$$\begin{aligned}&\mathcal{P}^{(1)}\delta'_{\mathrm{tr}}\lambda_\chi = \hat{H}G'_{\mathrm{tr}} + \lambda_{\mathcal{P}}\{\mathcal{P}^{(1)}, G'_{\mathrm{tr}}\} + O(\Omega^2),\\ &\delta'_{\mathrm{tr}}\lambda_{\mathcal{P}} = \delta_{\mathrm{tr}}\lambda_{\mathcal{P}}|_{I=0} = \delta_{\mathrm{tr}J}\lambda_{\mathcal{P}} + O(\Omega),\end{aligned} \tag{42}$$

which is similar to equation (29). Equation (42) implies the following equation for the local functions $K_{i|a}, a = i+1, \ldots, \aleph_{\mathcal{P}}$:

$$\sum_{i=1}^{\aleph_{\mathcal{P}}-1}\sum_{a=i+1}^{\aleph_{\mathcal{P}}}(\mathcal{P}^{(i|a)}\hat{H}K_{i|a} + K_{i|a}\mathcal{P}^{(i+1|a)}) = \mathcal{P}^{(1)}\delta'_{\mathrm{tr}}\lambda_{\mathcal{P}} + O(\Omega^2).$$

Considering this equation on the constraint surface $\Omega^{(\ldots\aleph_{\mathcal{P}}-1)} = 0$, we obtain that

$$K_{\aleph_{\mathcal{P}}-1|\aleph_{\mathcal{P}}}\mathcal{P}^{(\aleph_{\mathcal{P}}|\aleph_{\mathcal{P}})} = O(\Omega^2) \Longrightarrow K_{\aleph_{\mathcal{P}}-1|\aleph_{\mathcal{P}}} = 0.$$

Substituting the expression for $K_{\aleph_{\mathcal{P}}-1|\aleph_{\mathcal{P}}}$ into equation (42), and considering the resulting equation on the constraint surface $\Omega^{(\ldots\aleph_{\mathcal{P}}-2)} = 0$, we obtain $K_{\aleph_{\mathcal{P}}-2|\aleph_{\mathcal{P}}} = 0$, and so on. We thus see that all $K_{i|a} = 0, a = i+1, \ldots, \aleph_{\mathcal{P}}$, and therefore

$$G_{\mathrm{tr}} = O(I^2). \tag{43}$$

It then follows from equation (42)

$$\mathcal{P}^{(1)}\delta'_{\mathrm{tr}}\lambda_{\mathcal{P}} = O(\Omega^2) \Longrightarrow \delta'_{\mathrm{tr}}\lambda_{\mathcal{P}} = O(\Omega) \Longrightarrow \delta_{\mathrm{tr}}\lambda_{\mathcal{P}} = O(I).$$

By construction, the transformation $\delta_{\mathrm{tr}J}$ is similar to δ_J. Therefore, relation (31) holds true for this transformation and implies that

$$\delta_{\mathrm{tr}J}\vartheta = \{\vartheta, G'_{\mathrm{tr}}\} + O(\Omega) = O(\Omega), \qquad \delta_{\mathrm{tr}J}\lambda_U = O(\Omega).$$

Therefore,

$$\delta_{\mathrm{tr}}\boldsymbol{\vartheta} = O(I). \tag{44}$$

Relations (43) and (44) prove that the symmetry $\delta_{\mathrm{tr}}\boldsymbol{\vartheta}, G_{\mathrm{tr}}$ is trivial.

6. Physical functions

In spite of the fact that there exists a functional arbitrariness in solutions to equations of motion of a gauge theory, physics can be described by such theories. To ensure the independence of the physical quantities from the arbitrariness inherent to solutions of a gauge theory, one imposes limitations on the possible form of physical functions that describe the physical quantities. First of all, we recall the general understanding that physics can be described in terms of gauge theories [6]. Let the time evolution of a classical system be given by genuine trajectories $\kappa(t)$ in the configuration space. The latter are solutions to the equations of motion of the theory. On the other hand, the state of the classical system at any given time instant t is characterized by the set $\kappa^{[]}(t) = (\kappa^{[l]}(t), l \geqslant 0)$, at this time instant, i.e., by a point in the jet space. The trajectory in the configuration space creates a trajectory in the jet space. The latter trajectory can be called the trajectory of system states. We call two trajectories in the configuration space intersecting if the corresponding trajectories in the jet space intersect at a given time instant. Using such a terminology and the results of section 3, we can say that intersecting trajectories do exist in gauge theories. On the other hand, we believe that for classical systems, we can introduce the notion of the system physical state at each time instant, such that there exists a causal evolution of the physical states in time. Namely, once a physical state is given at a certain time, at all other times the physical states are determined in a unique way. All the physical quantities are single-valued functions of the physical state at a given time instant. The physical state is completely determined as soon as all possible physical quantities are given in a certain time instant. Therefore, at first glance, there is a disagreement between the causal evolution of the physical states and the absence of the causal evolution of trajectories in the jet space for gauge theories. To eliminate this discrepancy and to be able to describe classical systems consistently with the use of gauge theories, we can resort to the following natural interpretation:

(a) physical states of a classical system and, therefore, all local physical quantities are uniquely determined by points of genuine trajectories in the jet space;
(b) all the functions that are used to describe physical quantities must coincide at equal-time points of intersecting genuine trajectories in the jet space.

Item (b) ensures independence of the physical quantities from the arbitrariness inherent to solutions of a gauge theory and reconciles item (a) with the causal development of the physical states in time. Item (b) imposes limitations on the possible form of these functions. The local functions that obey item (b) are called physical functions. Suppose the local functions $\mathcal{A}_{\rm ph}(\kappa^{[]})$ are physical. This implies that for two arbitrary genuine intersecting trajectories κ and κ' the equality

$$\mathcal{A}_{\rm ph}(\kappa^{[]}) = \mathcal{A}_{\rm ph}(\kappa'^{[]}) \tag{45}$$

holds at any time instant.

We consider local physical functions in the Hamiltonian formulation and in the special phase-space variables $\boldsymbol{\vartheta}$. Taking the equations of motion (4), (5), and $\Omega = 0$ into account, we can conclude that any physical local functions of the form $\mathcal{A}_{\rm ph}(\boldsymbol{\vartheta}^{[]})$ can be represented as

$$\mathcal{A}_{\rm ph}(\boldsymbol{\vartheta}^{[]}) = a_{\rm ph}\big(\omega, Q, \lambda_{\mathcal{P}}^{[]}\big) + O\left(\frac{\delta S}{\delta \boldsymbol{\vartheta}}\right).$$

It is now easy to establish restrictions on the functions $a_{\rm ph}$ that follow from condition (45) of physicality. For this, we recall that there exist two genuine trajectories $\boldsymbol{\vartheta}$ and $\boldsymbol{\vartheta}'$ intersecting at $t = 0$ such that at the time instant t they, having the same ω, differ only by the values of the variables Q and $\lambda_{\mathcal{P}}^{[l]}$. Namely,

$$\boldsymbol{\vartheta}(t) \Longrightarrow \big(Q, \lambda_{\mathcal{P}}^{[]}\big), \qquad \boldsymbol{\vartheta}'(t) \Longrightarrow \big(Q + \delta Q, \lambda_{\mathcal{P}}^{[]} + \delta\lambda_{\mathcal{P}}^{[]}\big), \tag{46}$$

where all the quantities $Q, \lambda_{\mathcal{P}}^{[]}, \delta Q$ and $\delta\lambda_{\mathcal{P}}^{[]}$ are arbitrary. The existence of such intersecting trajectories follows from the consideration in section 3. Relation (45) for two such intersecting trajectories implies the relation,

$$a_{\rm ph}\big(\omega(t), Q, \lambda_{\mathcal{P}}^{[]}\big) = a_{\rm ph}\big(\omega(t), Q + \delta Q, \lambda_{\mathcal{P}}^{[]} + \delta\lambda_{\mathcal{P}}^{[]}\big), \tag{47}$$

for the function $a_{\rm ph}$. Because of the arbitrariness of the quantities $Q, \lambda_{\mathcal{P}}^{[]}, \delta Q$ and $\delta\lambda_{\mathcal{P}}^{[]}$, we obtain from equation (47) that

$$\frac{\partial a_{\rm ph}}{\partial Q} = \frac{\partial a_{\rm ph}}{\partial \lambda^{[]}} = 0 \Longrightarrow a_{\rm ph} = a_{\rm ph}(\omega). \tag{48}$$

Therefore, physical local functions of the form $\mathcal{A}_{\rm ph}(\boldsymbol{\vartheta}^{[]})$ can be represented as

$$\mathcal{A}_{\rm ph}(\boldsymbol{\vartheta}^{[]}) = a_{\rm ph}(\omega) + O\left(\frac{\delta S_{\rm H}}{\delta \boldsymbol{\vartheta}}\right). \tag{49}$$

In terms of the initial phase-space variables $\boldsymbol{\eta} = (\eta, \lambda), \eta = (q, p)$, any physical local functions of the form $A_{\rm ph}(\boldsymbol{\eta}^{[]})$ have the structure

$$A_{\rm ph}(\boldsymbol{\eta}^{[]}) = a_{\rm ph}(\eta) + O\left(\frac{\delta S_{\rm H}}{\delta \boldsymbol{\eta}}\right), \qquad \{a_{\rm ph}, \chi\} = O(\Phi). \tag{50}$$

Indeed, taking into account that the set of constraints $\mathcal{P}$ is equivalent to all FCC $\chi(\eta)$ in the initial phase-space variables, and that the set of constraints Ω is equivalent to all the initial constraints $\Phi(\eta)$, one can justify the second condition (50). We are going to call conditions (50) the physicality condition in the Hamiltonian sense. It is precisely in this sense one has to understand the usual assertion that physical functions must commute with first-class constraints on extremals. In fact, these conditions of physicality are those which are usually called the Dirac conjecture.

On the other hand, it is known that physical functions must be gauge invariant on the extremals (see, e.g., [6]). Let $\delta_\nu \boldsymbol{\eta}$ be a gauge symmetry in the Hamiltonian formulation. Then, the gauge variations of the LF $A_{\mathrm{ph}}(\boldsymbol{\eta}^{[]})$ must be proportional to extremals,

$$\delta_\nu A_{\mathrm{ph}}(\boldsymbol{\eta}^{[]}) = O\left(\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\eta}}\right). \tag{51}$$

Such a condition we call the physicality condition in the Lagrangian sense. Until now it was not clear whether two definitions (50) and (51) are equivalent. Below, we are going to demonstrate the equivalence of these two conditions for the general quadratic gauge theory.

Let a local function $A_{\mathrm{ph}}(\boldsymbol{\eta}^{[]})$ be physical in the Hamiltonian sense. Consider its gauge variation $\delta_\nu A_{\mathrm{ph}}$. Such a variation has the following form (having (50) in mind):

$$\delta_\nu A_{\mathrm{ph}} = \delta a_{\mathrm{ph}}(\eta) + O\left(\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\eta}}\right). \tag{52}$$

Here we have used the fact that gauge variations of extremals are proportional to extremals. Let us consider δa_{ph} taking into account (13) and (12). Then one easily sees that

$$\delta a_{\mathrm{ph}} = \{a_{\mathrm{ph}}, G\} = O(\{a_{\mathrm{ph}}, \chi\}) + O\left(\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\eta}}\right).$$

Taking into account (50), we obtain that gauge variations of physical functions are proportional to extremals, i.e., relation (51) holds.

Let now a local function $A_{\mathrm{ph}}(\boldsymbol{\eta}^{[]})$ be physical in the Lagrangian sense, i.e., they obey equation (51). One can always represent the function in the form

$$A_{\mathrm{ph}} = f\left(\eta, \lambda_{\mathcal{P}}^{[]}\right) + O\left(\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\eta}}\right).$$

Condition (51) implies

$$\{f, G\} + \sum_{m=0}^{m_{\max}} \frac{\partial f}{\partial \lambda_{\mathcal{P}}^{[m]}} \delta \lambda_{\mathcal{P}}^{[m]} = O\left(\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\eta}}\right). \tag{53}$$

Let us consider the terms containing the highest time-derivatives of the gauge parameters in the left-hand side of (53). Taking into account that $\delta \lambda_{\mathcal{P}}^{a} = \nu_a^{[a]}$, see (21), and the fact that G contains only the time derivatives $\nu_a^{[l]}$, $l < a$, such terms have the form

$$\sum_{a}^{\aleph_\chi} \frac{\partial f}{\partial \lambda_{\mathcal{P}}^{a[m_{\max}]}} \nu_a^{[a+m_{\max}]}.$$

These terms have to be proportional to the extremals, which implies

$$\frac{\partial f}{\partial \lambda_{\mathcal{P}}^{a[m_{\max}]}} = O\left(\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\eta}}\right).$$

Similarly, we can verify that the function f does not contain any λ on the extremals, i.e.,

$$f\left(\eta, \lambda_{\chi}^{[]}\right) = a(\eta) + O\left(\frac{\delta S_{\mathrm{H}}}{\delta \boldsymbol{\eta}}\right).$$

Therefore,

$$A_{\rm ph}(\boldsymbol{\eta}^{[]}) = a(\eta) + O\left(\frac{\delta S_{\rm H}}{\delta\boldsymbol{\eta}}\right). \tag{54}$$

Considering equation (51) for function (54), we obtain that

$$\{a, G\} = \sum_{i=1}^{\aleph_\chi}\sum_{b=i}^{\aleph_\chi}\{a, \mathcal{P}^{(i|b)}\}\nu_b^{[b-i]} = O\left(\frac{\delta S_{\rm H}}{\delta\boldsymbol{\eta}}\right),$$

which implies

$$\{a, \mathcal{P}^{(i|b)}\} = O\left(\frac{\delta S_{\rm H}}{\delta\boldsymbol{\eta}}\right) = O(\Phi) \Longrightarrow \{a, \chi\} = O(\Phi) \tag{55}$$

because all the $\nu_b^{[b-i]}$ are independent. This completes the proof of the equivalence of the two definitions of physical functions. Equations (54) and (55) indicate that the function $A_{\rm ph}(\boldsymbol{\eta}^{[]})$ is physical in the Hamiltonian sense.

7. Conclusion

We summarize below the main conclusions.

Any continuous symmetry transformation can be represented as a sum of three kinds of symmetries, a global symmetry, a gauge symmetry and a trivial symmetry. If the global part of a symmetry and the corresponding canonical charge are not identically zero, they do not vanish on the extremals. The determination of the canonical charge from the corresponding equation, and therefore the determination of the canonical part of a symmetry transformation, is ambiguous. However, we must understand that the ambiguity in the canonical part of a symmetry transformation is always a sum of a gauge transformation and a trivial transformation. The gauge part of a symmetry does not vanish on the extremals, but the gauge charge vanishes on the extremals. We emphasize that the gauge charge necessarily contains a part that is linear in the FCC, and the remaining part of the gauge charge is quadratic in the extremals. The trivial part of any symmetry vanishes on the extremals and the corresponding charge is quadratic in the extremals.

The reductions of global symmetry transformations to the extremals are global canonical symmetries of the physical action whose conserved charge is the reduction of the complete conserved charge to the extremals. Any global symmetry of the physical action is a global symmetry of the complete Hamiltonian action. The gauge transformations, taken on the extremals, only transform the nonphysical variables Q and $\lambda_{\mathcal{P}}$.

We can see that any gauge transformation can be represented in form (12) with an accuracy of a trivial transformation. This follows from the structure of arbitrary symmetry transformation presented above. Namely, as was demonstrated, any symmetry transformation whose charge vanishes on the extremals is a sum of a particular gauge transformation and of a trivial transformation. The gauge charge contains time derivatives of the gauge parameters whenever there exist secondary FCC.

Another assertion holds. We can see that the numbers of nonphysical variables both in Lagrangian and Hamiltonian formulations are respectively equal to the complete numbers of gauge parameters and their time derivatives that enter in the gauge transformations in these formulations. Indeed, in the Lagrangian formulation, the number of nonphysical coordinates coincides with the number of FCC in the Hamiltonian formulation (with the number of variables Q), and, therefore, coincides with the complete number of gauge parameters and their time derivatives that enter the gauge transformations of the coordinates in the Lagrangian formulation. In the Hamiltonian formulation, the nonphysical variables are both Q and λ_χ. At the same time, in this formulation, the gauge transformations of the Lagrange multipliers λ_χ contain an additional time derivative in comparison with the gauge transformations of the coordinates in the Lagrangian formulation. The number of λ_χ is equal to the number of primary FCC and, therefore, is equal to the number of gauge parameters. A simple estimation confirms the above assertion.

The equivalence of two definitions of physicality was proved. One of them states that physical functions are gauge invariant on the extremals, and the other definition requires that physical functions commute with FCC (the Dirac conjecture). As to the Dirac conjecture, we have demonstrated that it follows from the fact established in section 3 that all the special variables Q are controllable by the undetermined Lagrange multipliers.

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Appendix

In this appendix, we prove the existence of the super special phase-space variables described in section 2. First we consider the term $Q^{(1)}A\mathcal{P}^{(2\ldots)}$ in the Hamiltonian (2). Let the momenta $\mathcal{P}^{(1)}$ and the corresponding coordinates $Q^{(1)}$ be labelled by Greek subscripts, while the momenta $\mathcal{P}^{(2\ldots)}$ and the corresponding coordinates $Q^{(2\ldots)}$ are labelled by Latin subscripts,

$$Q = \left(Q_\nu^{(1)}, Q_b^{(2\ldots)}\right), \qquad \mathcal{P} = \left(\mathcal{P}_\nu^{(1)}, \mathcal{P}_b^{(2\ldots)}\right).$$

We assume that the defect of the rectangular matrix $A^{\nu b}$ is equal to $\boldsymbol{a}$ (it is evident that[7] $[Q^{(1)}] - \boldsymbol{a} \leqslant [Q^{(2\ldots)}]$). Then, there are $\boldsymbol{a}$ nontrivial null vectors $z_{(\tilde{\alpha})}, \tilde{\alpha} = 1, \ldots \boldsymbol{a}$, of the matrix A such that $z^\nu_{(\tilde{\alpha})}A^{\nu b} = 0$. We construct a quadratic matrix

$$Z_\alpha^\nu = \left\| z^\nu_{(\tilde{\alpha})} z^\nu_{(\bar{\alpha})} \right\|, \qquad \alpha = (\tilde{\alpha}, \bar{\alpha}), \qquad [\bar{\alpha}] \leqslant [b],$$

where the vectors $z_{(\bar{\alpha})}$ guarantee the nonsingularity of the complete matrix Z. Such vectors always exist. We then perform the canonical transformation $(Q^{(1)}, \mathcal{P}^{(1)}) \to (Q'^{(1)}, \mathcal{P}'^{(1)})$, where $Q_\nu'^{(1)} Z_\alpha^\nu = Q_\alpha^{(1)}$. Such a canonical transformation can be performed with a generating function of the form

$$W = Q_\nu'^{(1)} Z_\alpha^\nu \mathcal{P}_\alpha^{(1)}. \tag{A.1}$$

We denote the notation

$$Q_\alpha'^{(1)} = \left(Q_{\tilde{\alpha}}'^{(1)} = Q_{\tilde{\alpha}}^{(1|1)}, Q_{\bar{\alpha}}'^{(1)} = \bar{Q}_{\bar{\alpha}}^{(1)}\right), \qquad \mathcal{P}_\alpha'^{(1)} = \left(\mathcal{P}_{\tilde{\alpha}}^{(1|1)}, \bar{\mathcal{P}}_{\bar{\alpha}}^{(1)}\right).$$

Therefore, the primary FCC now are $\mathcal{P}^{(1|1)}, \bar{\mathcal{P}}^{(1)}$ and the corresponding conjugate coordinates are $Q^{(1|1)}, \bar{Q}^{(1)}$. After the canonical transformation the total Hamiltonian $H^{(1)} = H_{\text{ph}} + H_{\text{non-ph}}$ becomes

$$\begin{aligned} H^{(1)} = H_{\text{ph}} &+ \bar{Q}^{(1)}A'\mathcal{P}^{(2\ldots)} + (Q^{(2\ldots)}B + \omega C)\mathcal{P}^{(2\ldots)} + \mathcal{P}^{(2\ldots)}D\mathcal{P}^{(2\ldots)} \\ &+ \mathcal{P}^{(2\ldots)}EU^{(2\ldots)} + U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}, \end{aligned} \tag{A.2}$$

[7] The following notation is used: suppose $F_a(\eta), a = 1, \ldots, n$ are some functions, then $[F]$ is the number of these functions, $[F] = n$. Note that the brackets [] are also used to denote time-derivatives ($q^{[l]} = (d_t)^l q$) and arguments of action functionals (e.g. $S[q]$).

where

$$A' = (A')^{\bar{\nu}b} = Z^{\bar{\nu}}_{\alpha} A^{\alpha b}, \quad \text{rank } A' = \max = [\bar{Q}^{(1)}], \qquad b = (\bar{\mu}, \bar{b}), \quad \det(A')^{\bar{\nu}\bar{\mu}} \neq 0,$$

and $\lambda_1 \mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)}$ denotes the terms proportional to the primary FCC. At the same time, the functions λ_1 and $\bar{\lambda}$ absorb the time derivative of the generating function (A.1). We note that the coordinates $Q^{(1|1)}$ do not enter the Hamiltonian $H^{(1)}$ (in fact, that was one of the aims of the above canonical transformation) and therefore, the consistency conditions for the constraints $\mathcal{P}^{(1|1)}$ do not yield any new constraints,

$$\{\mathcal{P}^{(1|1)}, H^{(1)}\} \equiv 0.$$

We consider the consistency conditions for the primary FCC $\bar{\mathcal{P}}^{(1)}$,

$$\{\bar{\mathcal{P}}^{(1)}, H^{(1)}\} = -A'\mathcal{P}^{(2\ldots)} = 0.$$

Because the rank of the matrix A' is maximal, the combinations $A'\mathcal{P}^{(2\ldots)}$ of the secondary FCC are independent. We can choose them as new momenta $\mathcal{P}'^{(2)}$ which are now second-stage FCC. For this, we perform a canonical transformation $(Q^{(2\ldots)}, \mathcal{P}^{(2\ldots)}) \to (Q'^{(2\ldots)}, \mathcal{P}'^{(2\ldots)})$ with the generating function

$$W = Q'^{(2)} A' \mathcal{P}^{(2\ldots)} + Q'^{(3\ldots)} A'' \mathcal{P}^{(2\ldots)}. \tag{A.3}$$

Here, the rectangular matrix A'' is chosen such that the quadratic matrix $\Lambda = \|A'A''\|$ is invertible, $\det \Lambda \neq 0$. Therefore, the new variables are

$$\mathcal{P}'^{(2\ldots)} = (\mathcal{P}'^{(2)}, \mathcal{P}'^{(3\ldots)}), \qquad Q'^{(2\ldots)} = (Q'^{(2)}, Q'^{(3\ldots)}),$$
$$\mathcal{P}'^{(2)} = A'\mathcal{P}^{(2\ldots)}, \qquad \mathcal{P}'^{(3\ldots)} = (A''\mathcal{P}^{(2\ldots)}), \qquad Q'^{(2\ldots)} = Q^{(2\ldots)}\Lambda^{-1}.$$

In terms of the new variables the Hamiltonian (A.2) is

$$\begin{aligned} H^{(1)} = H_{\text{ph}} &+ \bar{Q}^{(1)}\mathcal{P}'^{(2)} + (Q'^{(2\ldots)}B' + \omega C')\mathcal{P}'^{(2\ldots)} + \mathcal{P}'^{(2\ldots)}D'\mathcal{P}'^{(2\ldots)} \\ &+ \mathcal{P}'^{(2\ldots)}E'U^{(2\ldots)} + U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}. \end{aligned} \tag{A.4}$$

The matrices B', C', D' and E' differ from B, C, D and E because of the change of the variables and, at the same time, they absorb the time derivative of the generating function (A.1). We note that the latter derivative does not modify the term $Q'^{(1)}\mathcal{P}'^{(2)}$.

Explicitly separating the terms proportional to $\mathcal{P}'^{(2)}$ in equations (A.4) and omitting all the primes, we obtain

$$\begin{aligned} H^{(1)} = H_{\text{ph}} &+ (\bar{Q}^{(1)} + \Sigma_q S_q + \Sigma_p S_p)\mathcal{P}^{(2)} + (Q^{(2\ldots)}B + \omega C)\mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)}D\mathcal{P}^{(3\ldots)} \\ &+ \mathcal{P}^{(3\ldots)}EU^{(2\ldots)} + U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}, \end{aligned} \tag{A.5}$$

where $\Sigma = (\Sigma_q, \Sigma_p)$ is the set of all the phase-space variables except $Q^{(1|1)}, \bar{Q}^{(1)}$ and $\mathcal{P}^{(1|1)}, \bar{\mathcal{P}}^{(1)}$, while S_q, S_p, B, C, D, E, and F are some matrices.

We now perform a canonical transformation (we do not transform the variables $Q^{(1|1)}, \mathcal{P}^{(1|1)}$) with the generating function W,

$$W = \bar{\mathcal{P}}'^{(1)}(\bar{Q}^{(1)} + \Sigma_q S_q + \Sigma'_p S_p) + \Sigma'_p \Sigma_q,$$

which yields

$$\bar{\mathcal{P}}'^{(1)} = \bar{\mathcal{P}}^{(1)}, \qquad \bar{Q}'^{(1)} = \bar{Q}^{(1)} + \Sigma_q S_q + \Sigma_p S_p + O(\bar{\mathcal{P}}^{(1)}), \qquad \Sigma' = \Sigma + O(\bar{\mathcal{P}}^{(1)}).$$

In terms of the new variables, the Hamiltonian (A.5) takes the form

$$\begin{aligned} H^{(1)} = H_{\text{ph}} &+ \bar{Q}^{(1)}\mathcal{P}^{(2)} + (Q^{(2\ldots)}B + \omega C)\mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)}D\mathcal{P}^{(3\ldots)} \\ &+ \mathcal{P}^{(3\ldots)}EU^{(2\ldots)} + U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}, \end{aligned} \tag{A.6}$$

where B, C, D, E and F are some matrices, the primes are omitted and redefined functions $\lambda_{\mathcal{P}}$ absorb time derivative of the generating function.

At this stage of the procedure, we consider the term $Q^{(2)}B\mathcal{P}^{(3\ldots)}$ in the Hamiltonian (A.6). Let the variables $\bar{Q}^{(1)}, \bar{\mathcal{P}}^{(1)}; Q^{(2)}, \mathcal{P}^{(2)}$ be numbered by Greek subscripts, and the variables $Q^{(3\ldots)}, \mathcal{P}^{(3\ldots)}$ be labelled by Latin subscripts (in the general case the number of indices differs from that from the first stage of the procedure). We assume that the defect of the rectangular matrix $B^{\nu k}$ is equal to $\boldsymbol{b}$ (obviously $[Q^{(2)}] - \boldsymbol{b} \leqslant [Q^{(3\ldots)}]$). Then, there are $\boldsymbol{b}$ nontrivial null vectors $\upsilon_{(\tilde{\alpha})}, \tilde{\alpha} = 1, \ldots, \boldsymbol{b}$ of the matrix B such that $\upsilon^{\nu}_{(\tilde{\alpha})}B^{\nu k} = 0$. We construct a quadratic matrix

$$V^{\nu\alpha} = \left\|\upsilon^{\nu}_{(\tilde{\alpha})}\upsilon^{\nu}_{(\bar{\alpha})}\right\|, \qquad \alpha = (\tilde{\alpha}, \bar{\alpha}), \qquad [\bar{\alpha}] \leqslant [k],$$

where the vectors $\upsilon_{(\bar{\alpha})}$ provide the nonsingularity of the complete matrix V. Such vectors always exist. Then we perform a canonical transformation

$$\bar{Q}^{(1)}, \bar{\mathcal{P}}^{(1)}; Q^{(2)}, \mathcal{P}^{(2)} \to \bar{Q}'^{(1)}, \bar{\mathcal{P}}'^{(1)}; Q'^{(2)}, \mathcal{P}'^{(2)}, \qquad Q'^{(2)} = \left(Q'^{(2)}_{\tilde{\alpha}} = Q^{(2|2)}_{\tilde{\alpha}}, Q'^{(2)}_{\bar{\alpha}} = \tilde{Q}^{(2)}_{\bar{\alpha}}\right),$$

with the generating function

$$W = \bar{Q}'^{(1)}V\bar{\mathcal{P}}^{(1)} + Q'^{(2)}V\mathcal{P}^{(2)}.$$

In terms of the new variables, the Hamiltonian (A.6) has the form

$$\begin{aligned} H^{(1)} = H_{\mathrm{ph}} &+ (\bar{Q}^{(1)} + Q^{(2)}\Delta)\mathcal{P}^{(2)} + \tilde{Q}^{(2)}B\mathcal{P}^{(3\ldots)} + (Q^{(3\ldots)}K + \omega C)\mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)}D\mathcal{P}^{(3\ldots)} \\ &+ \mathcal{P}^{(3\ldots)}EU^{(2\ldots)} + U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \bar{\lambda}\bar{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}, \end{aligned} \tag{A.7}$$

where $\Delta = \frac{\partial V}{\partial t}V^{-1}$, B, K, C, D, E and F are some matrices (we omit all the primes and redefine λ). In particular,

$$\operatorname{rank} B = \max = [\tilde{Q}^{(2)}] \leqslant [\mathcal{P}^{(3\ldots)}].$$

We now perform a canonical transformation $\bar{Q}^{(1)}, \bar{\mathcal{P}}^{(1)}, Q^{(2)}, \mathcal{P}^{(2)} \to \bar{Q}'^{(1)}, \bar{\mathcal{P}}'^{(1)}, Q'^{(2)}, \mathcal{P}'^{(2)}$ with the generating function

$$W = (\bar{Q}^{(1)} + Q^{(2)}\Delta)\bar{\mathcal{P}}'^{(1)} + Q^{(2)}\mathcal{P}'^{(2)},$$

which yields

$$\bar{\mathcal{P}}'^{(1)} = \bar{\mathcal{P}}^{(1)}, \qquad \bar{Q}'^{(1)} = \bar{Q}^{(1)} + Q^{(2)}\Delta, \qquad \mathcal{P}'^{(2)} = \mathcal{P}^{(2)} - \Delta\bar{\mathcal{P}}^{(1)}, \qquad Q'^{(2)} = Q^{(2)}.$$

In terms of the new variables, the Hamiltonian (A.7) takes the form

$$\begin{aligned} H^{(1)} = H_{\mathrm{ph}} &+ Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{Q}^{(1)}\tilde{\mathcal{P}}^{(2)} + \tilde{Q}^{(2)}B\mathcal{P}^{(3\ldots)} \\ &+ (Q^{(3\ldots)}K + \omega C)\mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)}D\mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)}EU^{(2\ldots)} \\ &+ U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \lambda_2\mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}. \end{aligned}$$

The primes are omitted and $\lambda_{\mathcal{P}}$ are redefined. The time derivative of the generating function is absorbed by the term $\tilde{\lambda}\tilde{\mathcal{P}}^{(1)}$.

We note that the variables $Q^{(2|2)}$ do not enter the Hamiltonian and therefore, the consistency conditions for the constraints $\mathcal{P}^{(2|2)}$ do not yield any new constraints. In addition, we remark that at this stage of the procedure, the primary FCC are $\mathcal{P}^{(1|1)}, \mathcal{P}^{(1|2)}$ and $\tilde{\mathcal{P}}^{(1)}$.

We consider the consistency conditions for the second-stage FCC $\mathcal{P}^{(2)}$,

$$\{\tilde{\mathcal{P}}^{(2)}, H^{(1)}\} = -B\mathcal{P}^{(3\ldots)} = 0.$$

Because the rank of the matrix B is maximum, the combinations $B\mathcal{P}^{(3\ldots)}$ are independent. We can choose them as new momenta $\mathcal{P}'^{(3)}$ which are now third-stage FCC. For this, we perform a canonical transformation $(Q^{(3\ldots)}, \mathcal{P}^{(3\ldots)}) \to (Q'^{(3\ldots)}, \mathcal{P}'^{(3\ldots)})$ with the generating function

$$W = Q'^{(3)}B\mathcal{P}^{(3\ldots)} + Q'^{(4\ldots)}B'\mathcal{P}^{(3\ldots)}, \qquad Q^{(3\ldots)}_k = \left(Q^{(3)}_{\alpha'}, Q^{(4\ldots)}_{k'}\right).$$

The rectangular matrix B' is here chosen such that the quadratic matrix $\Lambda = \|BB'\|$ is invertible, $\det\Lambda \neq 0$. We thus obtain

$$\mathcal{P}'^{(3\ldots)} = \Lambda\mathcal{P} = \left(\mathcal{P}'^{(3)}_{\alpha'}, \mathcal{P}'^{(4\ldots)}_{k'}\right), \qquad \mathcal{P}'^{(3)}_{\alpha'} = B^{\alpha' k}\mathcal{P}^{(3\ldots)}_{k},$$
$$Q'^{(3\ldots)} = Q^{(3\ldots)}\Lambda^{-1} = \left(Q'^{(3)}_{\alpha'}, Q'^{(4\ldots)}_{k'}\right).$$

In terms of the new variables, the Hamiltonian (A.7) has the form

$$H^{(1)} = H_{\mathrm{ph}} + Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{Q}^{(1)}\tilde{\mathcal{P}}^{(2)} + \tilde{Q}^{(2)}\mathcal{P}^{(3)} + (Q^{(3\ldots)}K + \omega C)\mathcal{P}^{(3\ldots)} + \mathcal{P}^{(3\ldots)}D\mathcal{P}^{(3\ldots)}$$
$$+\mathcal{P}^{(3\ldots)}EU^{(2\ldots)} + U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \lambda_2\mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)},$$

where K, C, D, E, and F are some matrices and primes are omitted.

We separate terms proportional to $\mathcal{P}^{(3)}$ in this expression and obtain

$$\begin{aligned} H^{(1)} = H_{\mathrm{ph}} &+ Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{Q}^{(1)}\tilde{\mathcal{P}}^{(2)} + \mathcal{P}^{(3)}(\tilde{Q}^{(2)} + S_q\Xi_q + S_p\Xi_p) \\ &+ (Q^{(3\ldots)}K + \omega C)\mathcal{P}^{(4\ldots)} + \mathcal{P}^{(4\ldots)}D\mathcal{P}^{(4\ldots)} + \mathcal{P}^{(4\ldots)}EU^{(2\ldots)} \\ &+ U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \lambda_2\mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}, \end{aligned} \tag{A.8}$$

where S_q, S_p, K, C, D, E, and F are some matrices, and $\Xi = (\Xi_q, \Xi_p)$ is the set of all the phase-space variables, except for $Q^{(1|1)}, \mathcal{P}^{(1|1)}, Q^{(1|2)}, \mathcal{P}^{(1|2)}, \tilde{Q}^{(1)}, \tilde{\mathcal{P}}^{(1)}, Q^{(2|2)}, \mathcal{P}^{(2|2)}, \tilde{Q}^{(2)}$ and $\tilde{\mathcal{P}}^{(2)}$.

We now perform a canonical transformation (we do not transform the variables $Q^{(1|1)}, \mathcal{P}^{(1|1)}; Q^{(1|2)}, \mathcal{P}^{(1|2)}; \tilde{Q}^{(1)}, \tilde{\mathcal{P}}^{(1)}, Q^{(2|2)}, \mathcal{P}^{(2|2)}$) with the generating function

$$W = \tilde{\mathcal{P}}'^{(2)}(\tilde{Q}^{(2)} + S_q\Xi_q + S_p\Xi'_p) + \Xi'_p\Xi_q,$$

which yields

$$\tilde{\mathcal{P}}'^{(2)} = \tilde{\mathcal{P}}^{(2)}, \qquad \tilde{Q}'^{(2)} = \tilde{Q}^{(2)} + S_q\Xi_q + S_p\Xi_p + O(\tilde{\mathcal{P}}^{(2)}), \qquad \Xi' = \Xi + O(\tilde{\mathcal{P}}^{(2)}).$$

In terms of the new variables, the Hamiltonian (A.8) takes the form

$$\begin{aligned} H^{(1)} = H_{\mathrm{ph}} &+ Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{\mathcal{P}}^{(2)}(\tilde{Q}^{(1)} + R_q\Sigma_q + R_p\Sigma_p) + \tilde{Q}^{(2)}\mathcal{P}^{(3)} \\ &+ (Q^{(3\ldots)}K + \omega C)\mathcal{P}^{(4\ldots)} + \mathcal{P}^{(4\ldots)}D\mathcal{P}^{(4\ldots)} + \mathcal{P}^{(4\ldots)}EU^{(2\ldots)} \\ &+ U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \lambda_2\mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}, \end{aligned} \tag{A.9}$$

where $\Sigma = (\tilde{Q}^{(2)}, \tilde{\mathcal{P}}^{(2)}, \Xi) = (\Sigma_q, \Sigma_p)$ and R, K, C, D, E and F are some matrices, all the primes are omitted.

We perform a canonical transformation with the generating function

$$W = \tilde{\mathcal{P}}'^{(1)}(\tilde{Q}^{(1)} + R_q\Sigma_q + R_p\Sigma'_p) + \Sigma'_p\Sigma_q,$$

and obtain

$$\tilde{\mathcal{P}}'^{(1)} = \tilde{\mathcal{P}}^{(1)}, \qquad \tilde{Q}'^{(1)} = \tilde{Q}^{(1)} + R_q\Sigma_q + R_p\Sigma_p + O(\tilde{\mathcal{P}}^{(1)}), \qquad \Sigma' = \Sigma + O(\tilde{\mathcal{P}}^{(1)}).$$

In terms of the new variables, the Hamiltonian (A.9) takes the form

$$\begin{aligned} H^{(1)} = H_{\mathrm{ph}} &+ Q^{(1|2)}\mathcal{P}^{(2|2)} + \tilde{Q}^{(1)}\tilde{\mathcal{P}}^{(2)} + \tilde{Q}^{(2)}\mathcal{P}^{(3)} \\ &+ (Q^{(3\ldots)}K + \omega C)\mathcal{P}^{(4\ldots)} + \mathcal{P}^{(4\ldots)}D\mathcal{P}^{(4\ldots)} + \mathcal{P}^{(4\ldots)}EU^{(2\ldots)} \\ &+ U^{(2\ldots)}FU^{(2\ldots)} + \lambda_1\mathcal{P}^{(1|1)} + \lambda_2\mathcal{P}^{(1|2)} + \tilde{\lambda}\tilde{\mathcal{P}}^{(1)} + \lambda_U U^{(1)}, \end{aligned} \tag{A.10}$$

where K, C, D, E and F are some matrices, all the primes are omitted and $\lambda_{\mathcal{P}}$ are redefined.

Further transformations of the Hamiltonian (A.10) can be done using the same kind of canonical transformations as those used before. At the end of the procedure, we obtain form (3) for the non-physical part of the total Hamiltonian.

We emphasize some important facts related to the canonical transformation that was performed to reduce the total Hamiltonian to the form (3).

First, we note that the final variables ω, Q, Ω, where $\Omega = (\mathcal{P}, U)$ (superspecial phase-space variables) still remain special phase-space canonical variables ϑ and possess all the corresponding properties of such variables. Let the final superspecial phase-space canonical variables be labelled by primes while the initial special phase-space variables are without primes. We can see that

$$\mathcal{P}' = T\mathcal{P}, \qquad \mathcal{P}^{(1)\prime} = T^{(1)}\mathcal{P}^{(1)}, \qquad U' = U + O(\mathcal{P}), \qquad U^{(1)\prime} = U^{(1)},$$

such that $\mathcal{P}'$ are FCC, $\mathcal{P}^{(1)\prime}$ are primary FCC, U' are SCC, and $U^{(1)\prime}$ are primary SCC. The physical variables do not change on the constraint surface, $\omega \to \omega' = \omega + O(\mathcal{P})$. We emphasize that the superspecial variables $\mathcal{P}^{(i|a)}$ coincide with the FCC $\chi^{(i|a)}$ in the orthogonal constraint basis introduced in [10]. In the general nonquadratic theory, the relation is

$$\chi^{(i|a)} = \mathcal{P}^{(i|a)} + O(\vartheta\Omega). \tag{A.11}$$

We can also see that in the superspecial phase-space variables, the non-physical part of the Hamiltonian action can be written as

$$S_{\text{non-ph}} = \int \left[\mathcal{P}\hat{\Lambda}\mathcal{Q} + \sum_{i=1}^{\aleph_\chi} \mathcal{P}^{(i|i)}\dot{Q}^{(i|i)} + \mathcal{U}\hat{B}\mathcal{U} \right] \mathrm{d}t, \tag{A.12}$$

where $\hat{\Lambda}$ and $\hat{B}$ are first-order differential matrix operators and

$$\mathcal{Q} = \left(\lambda_{\mathcal{P}}^{a}, Q^{(i|a)}, i = 1, \ldots, a-1, a = 1, \ldots, \aleph_\chi\right), \qquad \mathcal{U} = (\lambda_U, U).$$

It is important that $[\mathcal{Q}] = [\mathcal{P}]$ because $[\lambda_{\mathcal{P}}] = [\mathcal{P}^{(1)}]$.

We can see that there are local operators $\hat{\Lambda}^{-1}$ and $\hat{B}^{-1}$ such that $\hat{\Lambda}\hat{\Lambda}^{-1} = \hat{\Lambda}^{-1}\hat{\Lambda} = 1$, $\hat{B}\hat{B}^{-1} = \hat{B}^{-1}\hat{B} = 1$. This assertion can be derived from the fact that by the construction of the special phase-space variables, the Hamiltonian equations of motion have the unique solution $\mathcal{P} = 0$ and $\mathcal{U} = 0$. Therefore, the equations

$$\frac{\delta S_{\mathrm{H}}}{\delta \mathcal{Q}} = 0 \Longrightarrow \hat{\Lambda}^{T}\mathcal{P} = 0, \qquad \frac{\delta S_{\mathrm{H}}}{\delta \mathcal{U}} = 0 \Longrightarrow \hat{B}\mathcal{U} = 0 \tag{A.13}$$

must have only the solution $\mathcal{P} = 0$ and $\mathcal{U} = 0$. We represent $\hat{\Lambda}$ as

$$\hat{\Lambda} = \Lambda\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = a\frac{\mathrm{d}}{\mathrm{d}t} + b,$$

where a and b are some constant matrices, and consider the solutions of the form $\mathcal{P}(t) = \mathrm{e}^{-Et}\mathcal{P}(0)$, where E is a complex number. We obtain that $\Lambda^{T}(E)\mathcal{P}(0) = 0$. The existence of the unique solution $\mathcal{P}(0) = 0$ implies

$$\forall E : \det \Lambda(E) \neq 0. \tag{A.14}$$

On the other hand, $\det \Lambda(E)$ is a polynomial of E. Because of (A.14), such a polynomial has no roots. That means that $\det \Lambda(E) = \text{const} = c$. In turn, this implies that

$$\Lambda^{-1}(E) = \frac{1}{c}\Delta(E),$$

where $\Delta(E)$ are the corresponding minors of the matrix $\Lambda(E)$. The latter minors are finite-order polynomials in E. Therefore, the operator

$$\hat{\Lambda}^{-1} = \frac{1}{c}\Delta\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$$

is a local operator. We can similarly prove the existence of the local operator $\hat{B}^{-1}$ (for this, it is convenient to reduce the Hamiltonian $H_{\mathrm{SCC}}^{(1)}$ to a canonical form, see below).

In the same manner, we can demonstrate that there is a choice of superspecial phase-space variables that already include the variables U and which significantly simplify the Hamiltonian $H^{(1)}_{\mathrm{SCC}}$. Namely, for such a choice the variables U have the structure $U = (V; u)$, where both V and u are sets of pairs of conjugate coordinates and momenta. The variables from these sets are divided into groups according to the stages of the Dirac procedure and organized in chains (labelled by the index a). The variables V consist of coordinates Θ and conjugate momenta Π, namely,

$$V = \left(\Theta^{(i|2a)}_{\mu_a}, \Theta^{(i|2a+1)}_{\nu_a,s}; \Pi^{(i|2a)}_{\mu_a}, \Pi^{(i|2a+1)}_{\nu_a,s}\right), \qquad 1 \leqslant a \leqslant \aleph_\varphi/2,$$
$$i = 1, \ldots, a, \quad s = 1, 2, \quad u = \left(u^{(1)}_{\zeta,s}, u^{(2a+1)}_{\nu_a,s}\right).$$

The variables $u^{(1)}$ are primary constraints (first-stage constraints); the variables $u^{(2a+1)}$ are $2a+1$-stage constraints; the variables $\Pi^{(i|2a)}$ and $\Pi^{(i|2a+1)}$ are i-stage constraints; the variables $\Theta^{(i|2a)}$ are $2a - (i-1)$-stage constraints; the variables $\Theta^{(i|2a+1)}$ are $2a+1-(i-1)$-stage constraints.

The variables are divided into even and odd chains. Variables in even chains (labelled by $2a$) are labelled by the index μ_a, variables in odd chains (labelled by $1, 2a+1$) are labelled by the index ζ, ν_a and by the index. The number of indices μ_a and ζ, ν_a can be equal to zero.

In terms of the variables V, u the Hamiltonian $H^{(1)}_{\mathrm{SCC}}$ becomes

$$H^{(1)}_{\mathrm{scc}} = h_{\mathrm{odd}} + h_{\mathrm{even}} + \lambda^{(1)} u^{(1)},$$
$$h_{\mathrm{even}} = \sum_{a=1} \left(\sum_{i=1}^{a-1} \Theta^{(i|2a)} \Pi^{(i+1|2a)} + \sigma_{2a} (\Theta^{(i|2a)})^2 + \lambda^{(2a)} \Pi^{(1|2a)} \right),$$
$$h_{\mathrm{odd}} = \sum_{a=1} \left(\sum_{i=1}^{a-1} \Theta^{(i|2a+1)} \Pi^{(i+1|2a+1)} + \sigma_{2a+1} \Theta^{(a|2a+1)} u^{(2a+1)} + \lambda^{(2a+1)} \Pi^{(1|2a+1)} \right),$$

where $\sigma \neq 0$ are some numbers. There is a summation over the indices μ, ν and ζ, in particular $\sigma_{2a}(\Theta^{(i|2a)})^2 = \sum_{\mu_a} \sigma_{2a,\mu_a} \left(\Theta^{(i|2a)}_{\mu_a}\right)^2$.

In the refined superspecial phase-space variables, the consistency conditions that start with the primary SCC require that all the corresponding Lagrange multipliers $\lambda^{(1)}$, $\lambda^{(2a)}$ and $\lambda^{(2a+1)}$ be zero. See the following scheme of constraint chains:

$$\begin{array}{lllllllllllll}
u^{(1)}_s & \to & \lambda^{(1)}_s \\
\Pi^{(1|2)} & \to & \Theta^{(1|2)} & \to & \lambda^{(2)} \\
\Pi^{(1|3)}_s & \to & \Pi^{(2|3)}_s & \to & u^{(3)}_s & \to & \Theta^{(2|3)}_s & \to & \Theta^{(1|3)}_s & \to & \lambda^{(3)}_s \\
\Pi^{(1|4)} & \to & \Pi^{(2|4)} & \to & \Theta^{(2|4)} & \to & \Theta^{(1|4)} & \to & \lambda^{(4)} \\
\vdots \\
\Pi^{(1|2a)} & \to & \cdots & \to & \Pi^{(a|2a)} & \to & \Theta^{(a|2a)} & \to & \cdots & \to & \Theta^{(1|2a)} & \to & \lambda^{(2a)} \\
\Pi^{(1|2a+1)}_s & \to & \cdots & \to & \Pi^{(a|2a+1)}_s & \to & u^{(2a+1)}_s & \to & \Theta^{(a|2a+1)}_s & \to & \cdots & \to & \Theta^{(1|2a+1)}_s \to \lambda^{(2a+1)}_s. \\
\vdots
\end{array}$$

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